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The Derivative and The Tangent Line

Rapid Learning Tutorial Series

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Learning Objectives

By completing this tutorial, you will:

- Learn the relationship between a secant line and a tangent line
- Understand the formulation and definition of a derivative
- Know how to find derivatives using the definition
- Apply the derivative to applications

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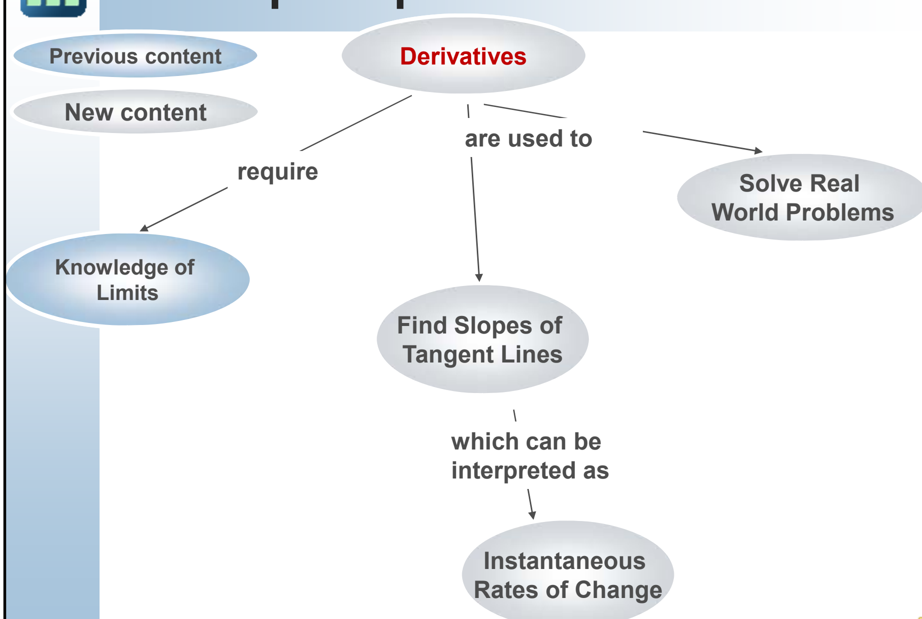
Concept Map



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What is a Tangent Line?

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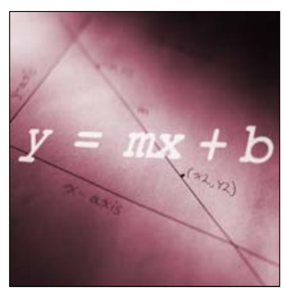
What is a Tangent Line - Outline

In this section, we will …

- Define the concept of a secant line
- Show the relationship between a secant line and a tangent line
- Relate the tangent line to instantaneous rates of change



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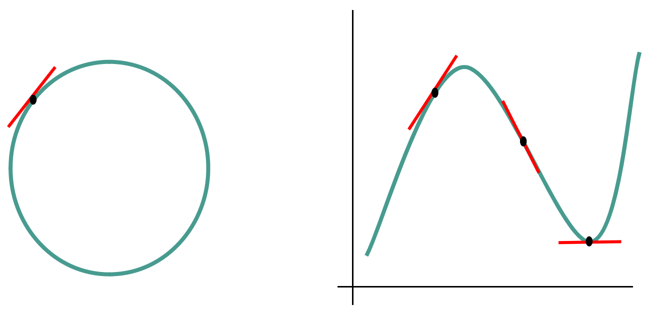
Definition – Tangent Line

A **tangent line** to a curve at a **given point** is a line that "barely touches" the curve at that point.



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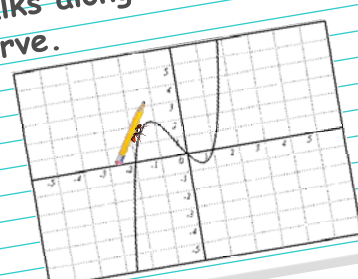
Tangent Line

Think about a **tangent line** as if it were a pencil balanced on the back of a bug as it walks along a given curve.



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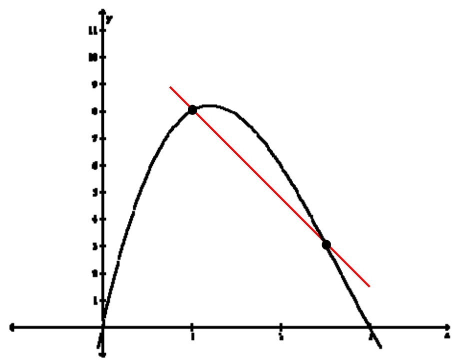
Example: Secant Line

The tangent line actually comes from another type of line called the **secant line**.

The **secant line** is a line that passes through **two points** on a function.



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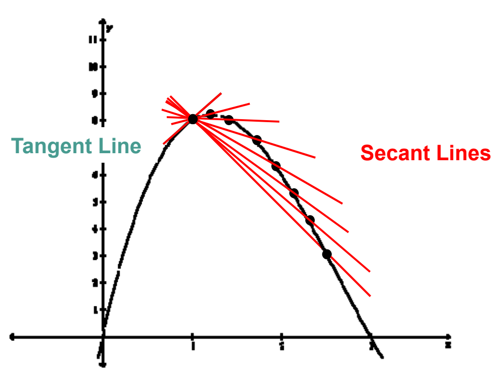
Example: Secant Line (Cont.)

As one of the points approaches the other, the secant line **converges** to the tangent line.



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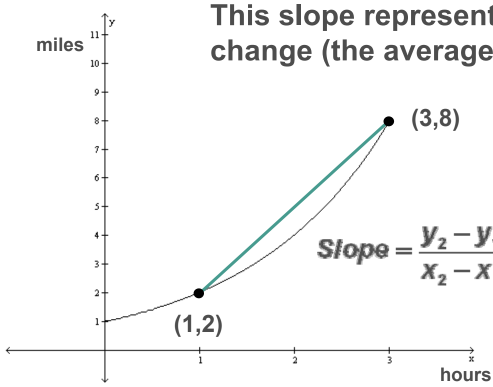
Example: Rate of Change

Suppose that the graph shown represents the distance walked after *x* hours.

Now consider the **slope** of a secant line between $x = 1$ and $x = 3$.

This slope represents the **average** rate of change (the average velocity in this case)



$$Slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3 \text{ miles/hr}$$

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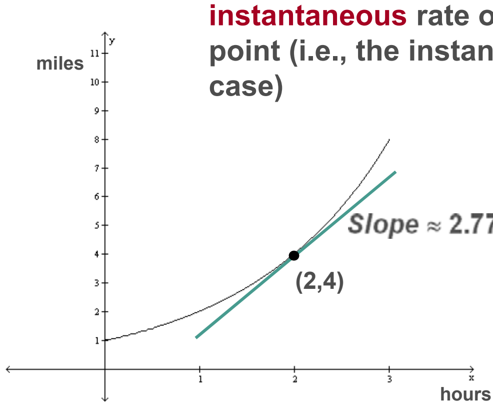
Example: Rate of Change (Cont.)

Suppose that the graph shown represents the distance walked after *x* hours.

The slope of a tangent line represents the **instantaneous** rate of change at the given point (i.e., the instantaneous velocity in this case)



$Slope \approx 2.77$ miles/hr

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Example: Velocity

Suppose that a rock is dropped from the top of the Eiffel Tower, 324 m above the ground. Find the velocity of the rock after 4 seconds.

The distance that an object has traveled after falling freely for t seconds is known to be $s(t) = 4.9t^2$ meters.

Here we are looking for the velocity at a single point in time ($t = 4$). We start by considering the average velocity.

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Example: Velocity (Cont.)

The average velocity is found by

$$\frac{\text{Distance Traveled}}{\text{Time It Takes}}$$

The average velocity over a short period of time from $t = 4$ to $t = 4.2$ is given by

$$\frac{s(4.2) - s(4)}{0.2} = \frac{4.9(4.2)^2 - 4.9(4)^2}{0.2}$$

$$= \frac{86.436 - 78.4}{0.2} = 40.18 \text{ meters/sec}$$

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Example: Velocity (Cont. 2)

The following table shows the average velocity for smaller and smaller time periods:

Time Interval	Average Velocity
$t = 4$ to $t = 4.1$	39.69

As the interval shortens, the average velocity appears to be converging towards 39.2 m/s, the instantaneous velocity

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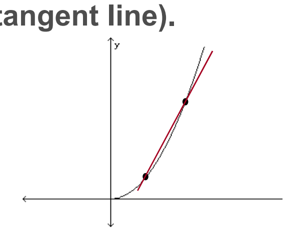
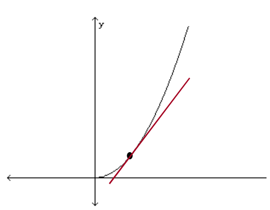
Example: Velocity (Cont. 3)

This example illustrates the difference between **average velocity** (slope of the secant line) and **instantaneous velocity** (slope of the tangent line).



Average Velocity



Instantaneous Velocity

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Derivatives

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Derivatives- Outline

In this section, we will …

- Learn about difference quotients
- Define the derivative
- Relate the concept of the derivative to tangent lines

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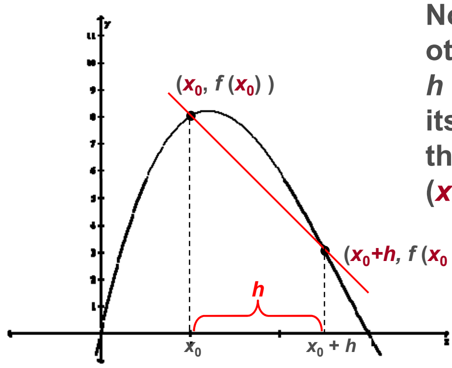
Introduction to Derivatives - 1

Consider again the example where one point on a graph of $f(x)$ approaches another. Suppose that the first point has a fixed x coordinate labeled x_0.

The point on the graph is then $(x_0, f(x_0))$



Now suppose that the other point on the graph is h units away. That makes its x coordinate $x_0 + h$ and the corresponding point is $(x_0 + h, f(x_0 + h))$

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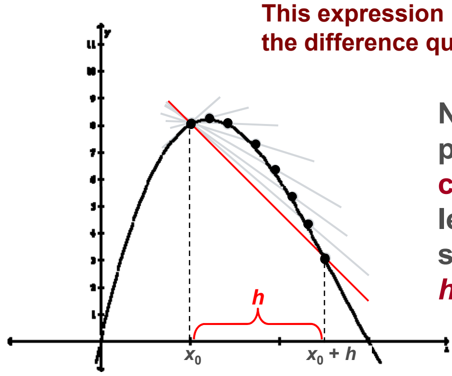
Introduction to Derivatives - 2

We can now write down a formula for the **slope** of the line in terms of x_0 and h.

$$Slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

This expression is called the difference quotient!



Now suppose that the point on the right **gets closer** to the point on the left. Note that this is the same thing as saying that $h \to 0$.

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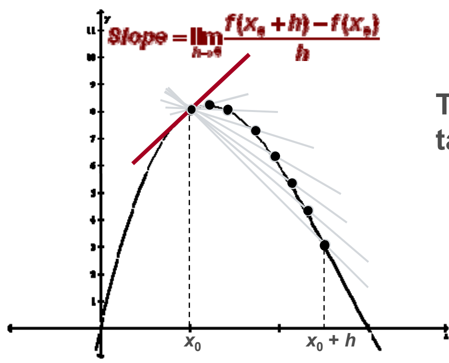
Introduction to Derivatives - 3

Since the secant line approaches the **tangent line** as the point on the right gets closer to the point on the left, the slope of the secant line approaches the **slope of the tangent line** as $h \to 0$.



Therefore, the slope of the tangent line is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Definition – Derivative

The **derivative** of a function $f(x)$ is defined to be the slope of the tangent line and is given by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is a notation for the derivative. There are several other common notations.

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Derivative Notations

The "prime" notation is the most common way to denote derivatives.
Other notations include the **differential operator notation** $D_x(f)$ and the **Leibnitz notation** $\frac{dy}{dx}$

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Example 1: Finding the Derivative

Consider the function $f(x) = x^2$.

The **derivative** is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0}(2x+h)$$

$$= 2x$$

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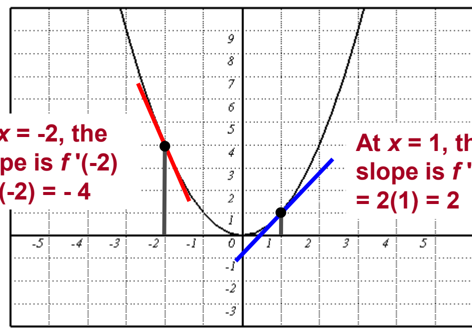
Example 1: Finding the Derivative (Cont.)

The function $f'(x) = 2x$ gives the **slope of the tangent line** at any given point on the graph of $f(x) = x^2$.

At $x = -2$, the slope is $f'(-2) = 2(-2) = -4$



At $x = 1$, the slope is $f'(1) = 2(1) = 2$

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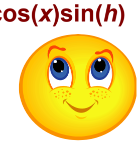
Example 2: Finding the Derivative

Consider the function $f(x) = \sin(x)$.

Remember that $\sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h)$



The **derivative** is given by

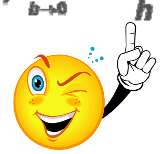
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

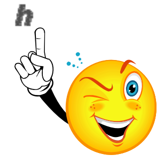
$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x) \cdot \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

Recall that this limit equals 0



And this one equals 1



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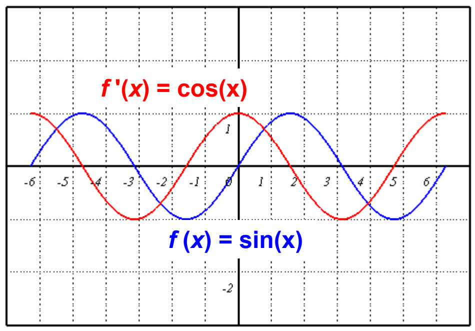
Example 2: Finding the Derivative (Cont.)

So the derivative is $f'(x) = \sin(x)\cdot(0) + \cos(x)\cdot(1)$

$= \cos(x)$



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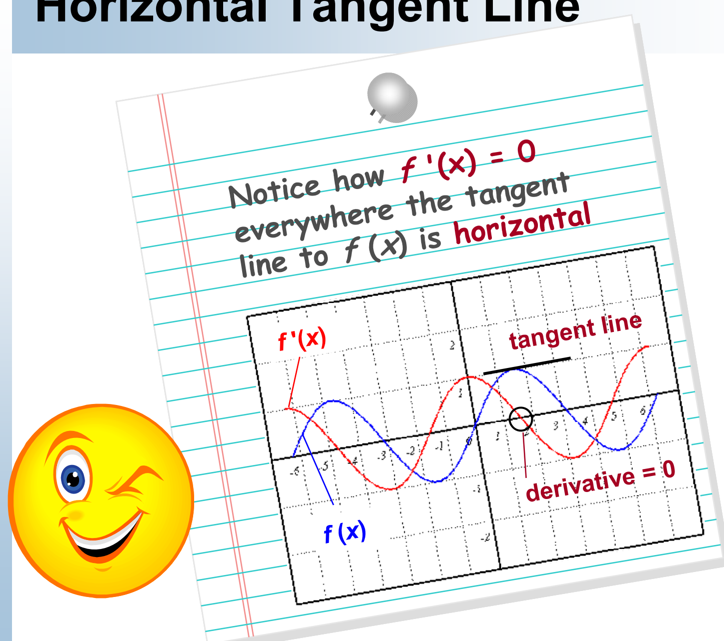
Horizontal Tangent Line



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Example 3: Equation of Tangent Line

Consider the function $f(x) = 2x^2 + 3$. We wish to find the **equation of the tangent line** at the point (2, 11).

Remember that to find the equation of a line, you need a point and a slope

The point is given. The slope of the tangent line at (2, 11) is $f'(2)$.

The derivative is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2+3-(2x^2+3)}{h}$$

$$= \lim_{h\to 0}\frac{2x^2+4xh+2h^2+3-2x^2-3}{h} = \lim_{h\to 0}\frac{4xh+2h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(4x+2h)}{h} = \lim_{h\to 0}(4x+2h) = 4x$$

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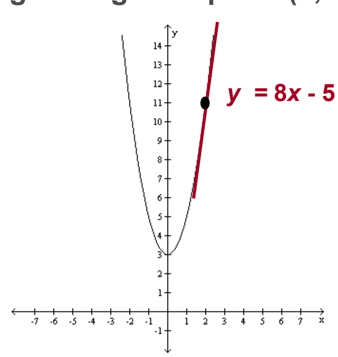
Example 3: Equation of Tangent Line (Cont.)

Consider the function $f(x) = 2x^2 + 3$. We wish to find the **equation of the tangent line** at the point (2, 11).

The **slope** of the tangent line is $f'(2) = 4(2) = 8$.

The equation of the line going through the point (2, 11) with a slope of 8 is:

$$y - y_1 = m(x - x_1)$$
$$y - 11 = 8(x - 2)$$
$$y - 11 = 8x - 16$$
$$y = 8x - 5$$



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There's an Easier Way

Right now, you are learning how the derivative is derived. When it comes to finding the derivative, there are often **shortcut formulas** that you can use that you will learn later.

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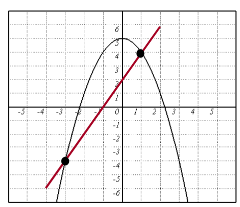
Check Yourself: True or False?

1) The following is an example of a secant line.



True.

2) The slope of the tangent line to $f(x) = x^3$ at $x = 1$ is given by $f(1)$.

False. **The slope of the tangent line is given by $f'(1)$.**

3) If the derivative equals zero at $x = c$, then the tangent line is horizontal at $x = c$.

True.

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Additional Examples

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Cost Function

Suppose that the cost $C(x)$ of producing x units of some product is given by $C(x) = 7x^2 + 100$.

The derivative is found to be $C'(x) = 14x$ which is interpreted as the **instantaneous rate of change** of the cost C with respect to the number of units x.

For example, $C'(10) = 14(10) = 140$ means that when 10 units are being produced, the cost is changing at the rate of $140 per unit.

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Population

Suppose that a population of a certain species of bird grows according to the model $P(x) = 3x^3$ where x represents time in months.

The derivative is found to be $P'(x) = 9x^2$ which is interpreted as the **instantaneous rate of change** of the population P with respect to time.

For example, $P'(6) = 9(6^2) = 324$ means that after 6 months, the population is increasing at the rate of 324 birds per month.

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Learning Summary

A **tangent line** to a curve at a given point is a line that "barely touches" the curve at that point.

The **secant line** approaches the **tangent line** when the two points get closer and closer together.

The **derivative** gives the slope of the tangent line.

The derivative is interpreted as the **instantaneous rate of change**.

The **derivative** is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Congratulations

You have successfully completed the core tutorial

The Derivative and The Tangent Line

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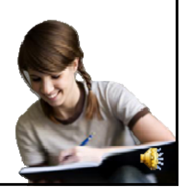
What's Next …

Step 1: Concepts – Core Tutorial (Just Completed)

→ **Step 2: Practice – Interactive Problem Drill**

Step 3: Recap – Super Review Cheat Sheet

Go for it!



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