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Limits

Rapid Learning Tutorial Series

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Learning Objectives

By completing this tutorial, you will:

- Learn the meaning of the limit of a function
- Learn how to approximate limits numerically
- Learn how to find a limit using algebra
- Learn about vertical asymptotes and removable discontinuities

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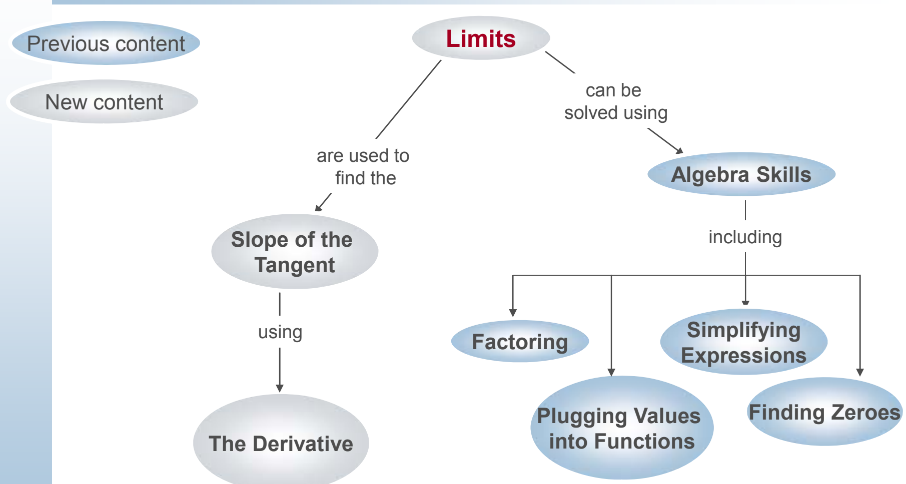
Concept Map



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What is a Limit?

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Outline – Limit Definition

This section will help you to:

- Express limits verbally
- Understand limit notation
- Understand what limits mean
- Apply limits to solve problems

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Limit Definition

The **limit of a function** as $x \rightarrow a$ is:

the output (*y*-value) the function *seems* like it will reach as the independent variable (*x*) gets closer and closer to the number *a*.

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Limit Notation

The notation for the limit of *a function of x*, as *x* approaches *a*, is:

$$\lim_{x \to a} f(x)$$

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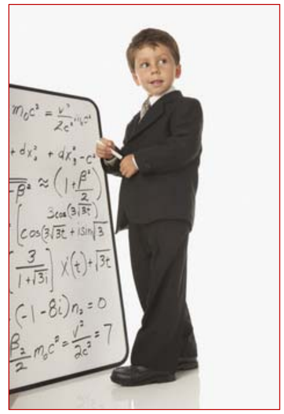
How to Read the Limit: Stop-and-think

How would you read this limit?

$$\lim_{t \to -7}\left(t^2 + 4t - 1\right)$$

Answer: ***The limit of t squared plus 4t minus one as t approaches negative 7.***



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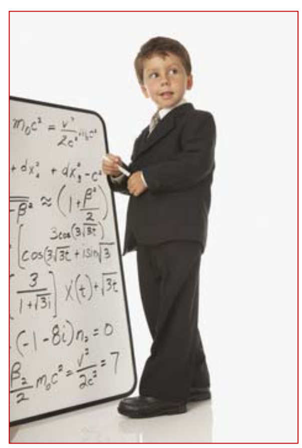
Meaning of Limit: Stop-and-think

What does this limit mean?

$$\lim_{t \to -7}\left(t^2 + 4t - 1\right)$$

Answer: It means, "As the independent variable, *t*, gets closer and closer to negative 7, what output value does the function, *t* squared plus 4*t* minus 1, seem to be approaching?"



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How are Limits Used?

Three common applications of limits:

- **The area of irregular shapes, using the method of exhaustion.**
- **Instantaneous velocity, using the difference quotient and the derivative.**
- **The sums of infinite series.**

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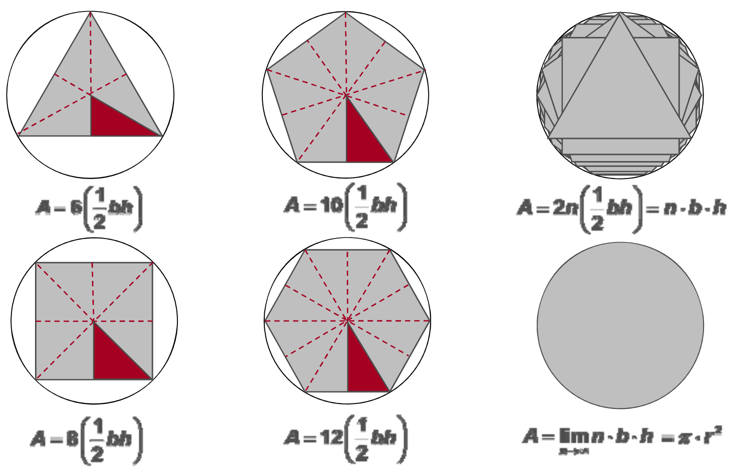
Areas of Irregular Shapes

As far back as the ancient Greeks, mathematicians have used the idea of limits to find the area of irregular shapes. The diagrams below depict the "method of exhaustion" as it is used to determine the area of a circle.



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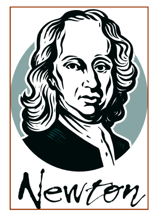
Instantaneous Velocity, Verbally

Velocity - The rate and direction of an object's change in position with respect to time.

Average Velocity - The change in an object's position divided by the amount of time that has elapsed. This ratio is known as a **difference quotient**.

Instantaneous Velocity - The velocity of an object at one exact instant, measured with a limit called the **derivative**.



Isaac Newton used derivatives to find instantaneous velocity

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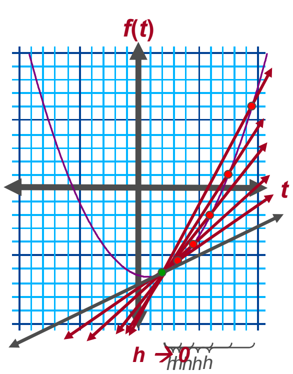
Graphing Instantaneous Velocity

To find the **instantaneous velocity** of an object, find the limit of the average velocity as the interval of elapsed time approaches zero.

$f(t)$ = position of an object at any time, t.
h = interval of elapsed time



The **average velocity** on an interval of time (h) is the slope of the line drawn between the interval end-points on the position function curve.

Let the elapsed time, h, get smaller and smaller, redrawing the line each time.

As h is getting smaller and smaller, we are observing the limit of "average velocity" as h approaches 0.

Now we have the slope of the line through just one point – the **instantaneous velocity**.

The limit, which gives us instantaneous velocity, is called the **derivative**.

$$v_{inst} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

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Definition: Infinite Series

Infinite Sequence: An unending list of numbers or other mathematical objects, called the **terms** of the sequence.

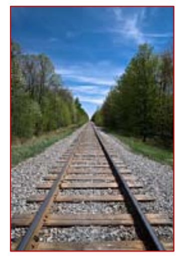
(16, 8, 4, 2, …)

(2, 6, 18, 54, …)

(3, 9, 15, 21, …)

Infinite Series: The sum of an unending sequence of terms.

Convergent Infinite Series: An infinite series that adds up to a finite number.



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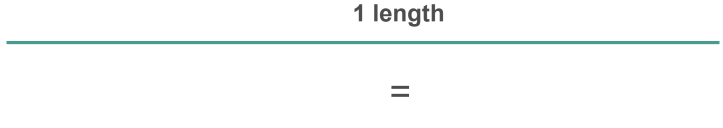
Infinite Series

The ancient Greek philosopher Zeno identified a paradox in the claim, "**That which is in locomotion must arrive at the half-way stage before it arrives at the goal.**" Zeno's paradox is that the object would never reach the goal.

1 length



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

The solution to the paradox, proposed by Georg Cantor in the 19th Century, uses the **limit** of a convergent infinite series.

In mathematical notation: $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^i}$

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Numerical Limits

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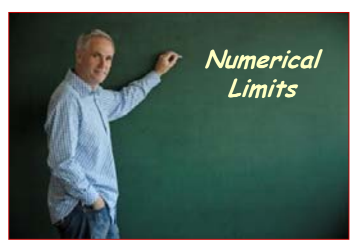
Numerical Limits - Outline

In this section you will:

- Find limits by substituting numbers
- Learn when the limit of a function does not exist



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Approximating Limits Numerically

To approximate a limit using numbers:

1. Identify the number the independent variable (*x*) is approaching. $x \rightarrow n$
2. Plug a number a slightly less than this value (n) in for the independent variable in the function.
3. Plug a number a slightly more than this value in for the independent variable in the function.
4. If the two values you get are very close to one another, then pick a number in between them; this is your guess for the limit of the function. If not, the limit does not exist.

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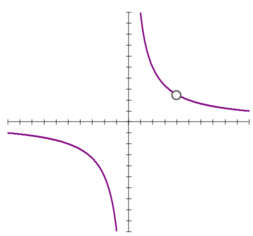
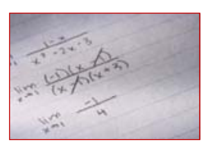
Example: Rational Function - Step 1

$$\lim_{x \to 4} \frac{10x - 40}{x^2 - 4x}$$



1. Identify the number that the independent variable is approaching.



x is the independent variable, and it is approaching 4.

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Note: What Numbers to Plug In?

For many functions, it is effective to plug in numbers 0.01 **below** and **above** the value x is approaching.

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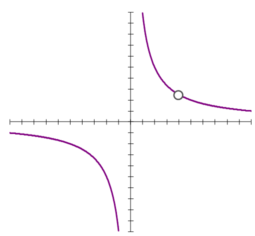
Example: Rational Function - Step 2

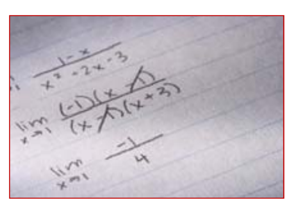
$$\lim_{x \to 4} \frac{10x - 40}{x^2 - 4x}$$



2. Since the independent variable is approaching 4, plug in a number a little ***less*** than 4 for x in the function.

Let $x = 3.99$

$$\frac{10(3.99) - 40}{(3.99)^2 - 4(3.99)} \approx 2.51$$



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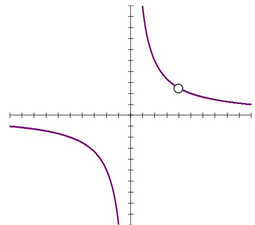
Example: Rational Function - Step 3

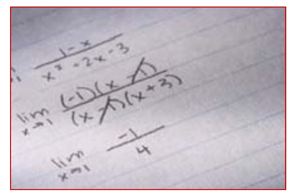
$$\lim_{x \to 4} \frac{10x - 40}{x^2 - 4x}$$



3. Since the independent variable is approaching 4, plug a number a little *more* than 4 in for x in the function.

Let x = 4.01

$$\frac{10(4.01) - 40}{(4.01)^2 - 4(4.01)} \approx 2.49$$



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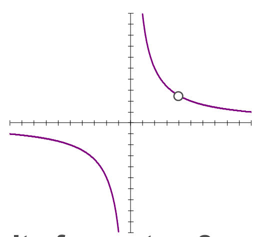
Example: Rational Function - Step 4

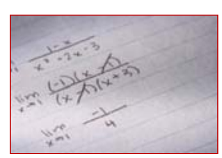
$$\lim_{x \to 4} \frac{10x - 40}{x^2 - 4x}$$



4. Analyze the results from step 2 and step 3 to see if they are close to each other. If so, pick a number in between them as an approximate answer for the limit.

Steps 2 and 3 yielded estimates of 2.51 and 2.49.

A reasonable approximation for the limit of this function is 2.5.



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Note: The Limit May Not Exist

In the last example, the two values were very close to each other. When this is not the case, we conclude **the limit does not exist.**

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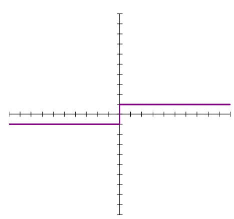
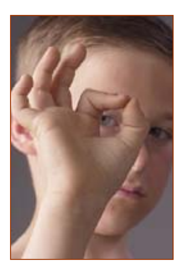
Example: Absolute Value - Step 1

$$\lim_{x \to 0} \frac{|x|}{x}$$



1. **Identify the number that the independent variable is approaching.**

x **is the independent variable, and it is approaching 0.**



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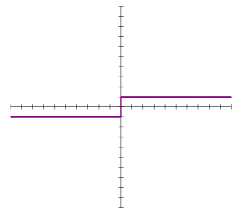
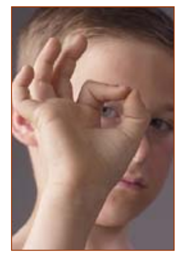
Example: Absolute Value - Step 2

$$\lim_{x \to 0} \frac{|x|}{x}$$



2 **Since *x* is approaching 0, plug in a number a little *less* than 0 for *x*.**



Let *x* = –0.01

$$\frac{|-0.01|}{-0.01} = -1$$

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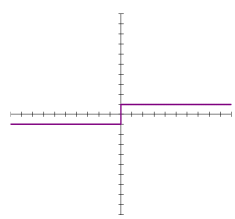
Example: Absolute Value - Step 3

$$\lim_{x \to 0} \frac{|x|}{x}$$



3 **Since *x* is approaching 0, plug in a number a little *more* than 0 for *x*.**

Let *x* = 0.01

$$\frac{|0.01|}{0.01} = 1$$

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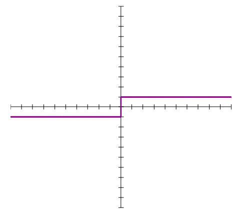
Example: Absolute Value - Step 4

$$\lim_{x \to 0} \frac{|x|}{x}$$



4. Analyze the results from steps 2 and 3 to determine whether they are close to each other.

The values we got were **–1** and **1**.

These values are not very close to each other!

This limit **does not exist**.

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Numerical Limits: Stop-and-think

$$\lim_{x \to 2} \frac{x^2}{x^2 - x - 2}$$

Does this limit exist?

Answer:

$$\frac{(1.99)^2}{(1.99)^2 - (1.99) - 2} \approx -132.44$$

$$\frac{(2.01)^2}{(2.01)^2 - (2.01) - 2} \approx 134.22$$

These values are not close to each other.
The limit does not exist.

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Finding Limits Algebraically

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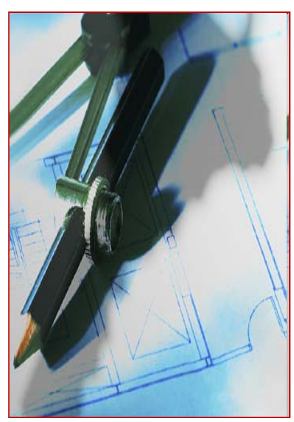
Finding Limits Algebraically - Outline

After completing this section you will be able to:

- Find the limit of a polynomial function
- Find the limit of a function at a vertical asymptote
- Find the limit of a function with a removable discontinuity



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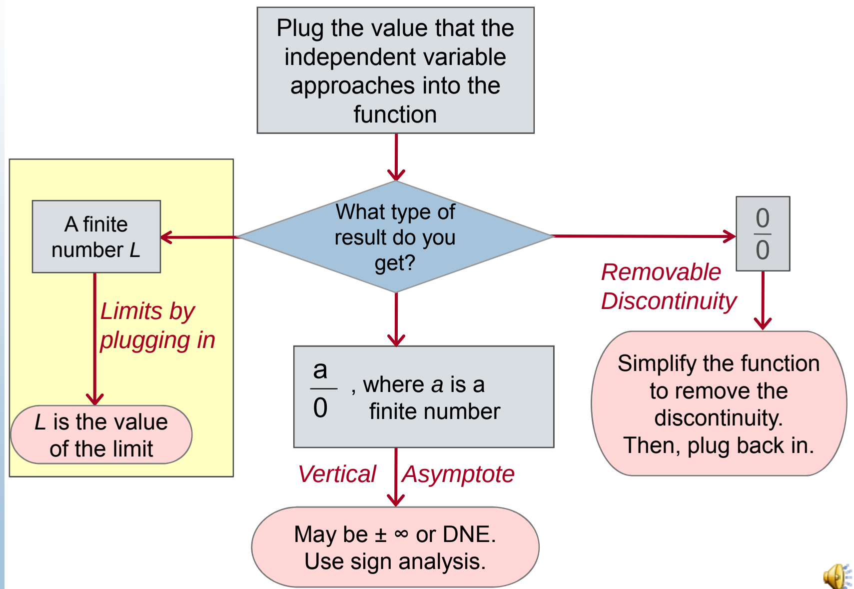
Finding Limits Algebraically



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Limits by Plugging In

Sometimes you can find a limit by plugging the value the independent variable approaches into the function. This is evident when the function is defined at the "plugged in" value.

1. **Plug the value the independent variable is approaching into the function.**
2. **If you get a finite number after simplifying, this number is the limit.**

   You are done!

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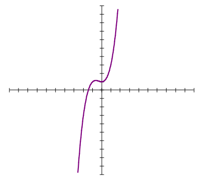
Example: Limits by Plugging In - Step 1

$$\lim_{x \to -5}\left(1 + x^2 + x^3\right)$$



1. Plug the value approached by the independent variable into the function.

$1 + (-5)^2 + (-5)^3$

$= 1 + 25 + (-125)$

= –99

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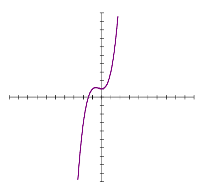
Example: Limits by Plugging In - Step 2

$$\lim_{x \to -5}\left(1 + x^2 + x^3\right)$$



2. What did you get when you plugged –5 into the function and simplified?

You got –99, which is a finite number.

The limit equals –99. Well done!

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Note: Polynomial Limits

In the last example, the function was a **polynomial**. You can always find the limit of a polynomial by plugging in.

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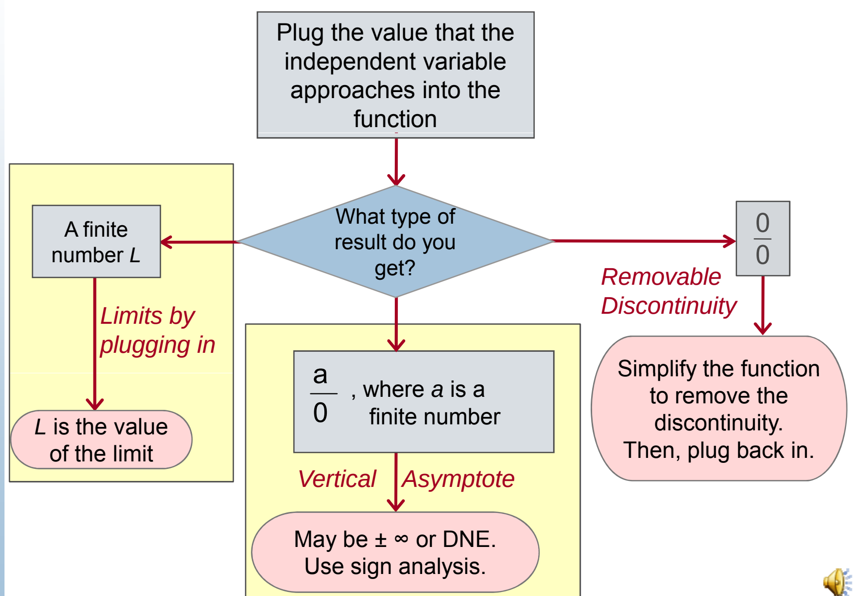
Finding Limits Algebraically



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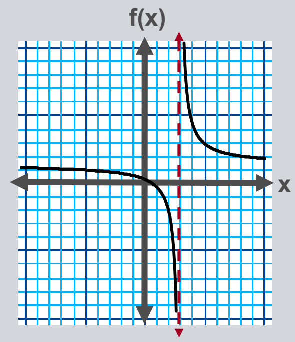
Vertical Asymptotes

A **vertical asymptote** is a line that the graph of a function gets closer and closer to but never reaches. The function is undefined at the *x*-value where the vertical asymptote occurs.



The dotted line above is a vertical asymptote at x = 3.

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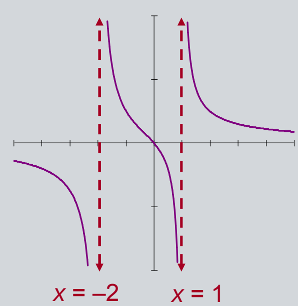
Vertical Asymptotes

A vertical asymptote occurs at a value of *x* when plugging the x-value in *f(x)* results in a **zero in the denominator** of *f(x)*.

$$f(x)=\frac{x}{(x-1)(x+2)}$$

When $x = 1$ or $x = -2$ the denominator = 0.



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Limits at a Vertical Asymptote

Here is how to find the limit at a vertical asymptote:

1. Plug the value the independent variable is approaching into the function.
2. If you get a fraction with a nonzero finite number in the numerator and with 0 in the denominator, then there is a vertical asymptote at the x-value.
3. To find the limit at the asymptote, plug numbers a little bit less and more than the *x*-value of the asymptote.
4. Both results positive → Limit = ∞

   Both results negative → Limit = – ∞

   One positive, one negative → Limit Does Not Exist (DNE)

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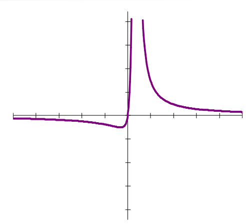
Example: Vertical Asymptote - Step 1

$$\lim_{x\to\frac{1}{3}}\frac{6x}{9x^2-6x+1}$$



1. Plug the value approached by the independent variable into the function.

$$\frac{6\left(\frac{1}{3}\right)}{9\left(\frac{1}{3}\right)^2-6\left(\frac{1}{3}\right)+1}=\frac{2}{1-2+1}=\frac{2}{0}$$

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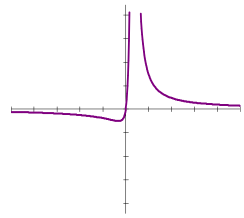
Example: Vertical Asymptote - Step 2

$$\lim_{x\to\frac{1}{3}}\frac{6x}{9x^2-6x+1}$$



2 Since the result is a finite number over 0, the function has a ***vertical asymptote*** at x equals one third.

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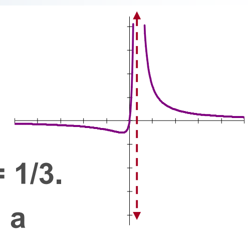
Example: Vertical Asymptote - Step 3

$$\lim_{x\to\frac{1}{3}}\frac{6x}{9x^2-6x+1}$$



3 There is a vertical asymptote at $x = 1/3$. Now plug numbers a little less and a little more than 1/3 in for x. We'll use the decimals 0.33 and 0.34.

$$\frac{6(0.33)}{9(0.33)^2-6(0.33)+1}=19{,}800$$

$$\frac{6(0.34)}{9(0.34)^2-6(0.34)+1}=5{,}100$$

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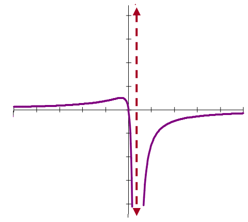
Example: Vertical Asymptote - Step 4

$$\lim_{x \to \frac{1}{3}} \frac{-6x}{9x^2 - 6x + 1}$$



4 The numbers we got were 19,800 and 5,100.

Both are large, positive numbers.

The limit is ∞.

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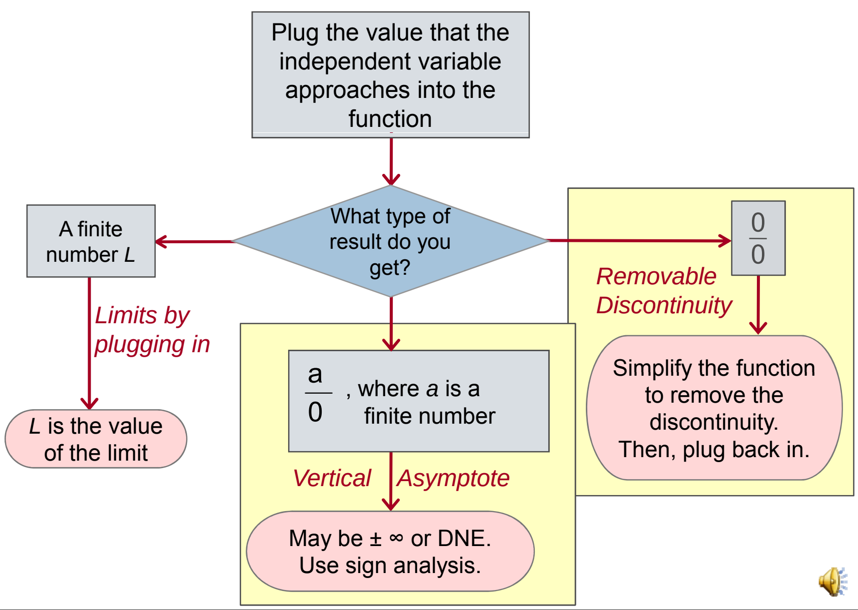
Finding Limits Algebraically



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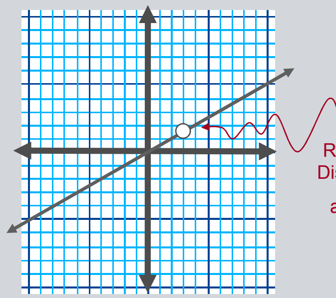
Removable Discontinuities

Removable Discontinuity: A single point that the graph of a function "skips." Usually the result of having a common factor in the numerator and denominator of the function, which can be cancelled.

$$f(x) = \frac{x^2 - 3x}{2x - 6}$$

can be simplified to:

$$\frac{x(x-3)}{2(x-3)} \rightarrow \frac{x}{2}$$



Removable Discontinuity at (3, 1.5)

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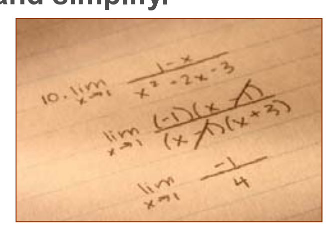
Limits at a Removable Discontinuity

Here is how to find the limit at a removable discontinuity:

1. Plug the value the independent variable is approaching into the function.
2. If you get a fraction with 0 in the numerator and in the denominator, then there is a removable discontinuity.
3. To remove the discontinuity, factor the numerator and denominator of the function and simplify.
4. When the function is simplified, try plugging in again. The number you get is the answer to the limit!



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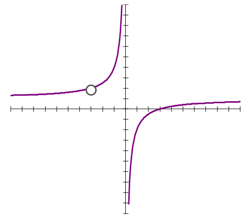
Example: Removable Discontinuity - Step 1

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 3x}$$



1. Plug the value approached by the independent variable into the function.

$$\frac{(-3)^2 - 9}{(-3)^2 + 3(-3)} = \frac{9 - 9}{9 + (-9)} = \frac{0}{0}$$

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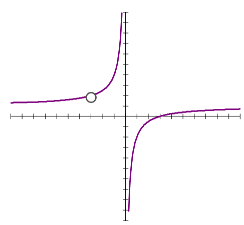
Example: Removable Discontinuity - Step 2

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 3x}$$



2. We got 0 divided by 0 when we plugged the value *x* approaches into the function.

   This means we have a ***removable discontinuity***.

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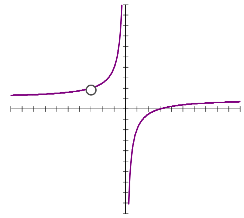
Example: Removable Discontinuity - Step 3

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 3x}$$



3. There is a removable discontinuity at $x = -3$.

Now we try to **factor** the numerator and denominator of the function, in order to simplify.

$$\frac{x^2 - 9}{x^2 + 3x} = \frac{(x-3)\cancel{(x+3)}}{x\cancel{(x+3)}} = \frac{x-3}{x}$$

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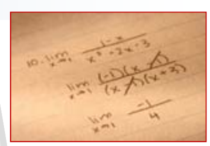
Note: Canceled Factors

Notice that the factor you canceled out was **(x + 3)**, which is what made the top and bottom **zero** in the first place.



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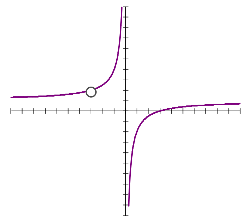
Example: Removable Discontinuity - Step 4

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 3x}$$



4 Having simplified the function by crossing out the factor (x + 3) in the numerator and denominator, plug the number x approaches, –3, into the simplified function.

$$\frac{(-3)-3}{(-3)} = \frac{-6}{-3} = 2$$

The limit equals 2.

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Learning Summary

Some limits can be solved just by **plugging in** the number x approaches.

You can approximate a limit by plugging in numbers a little less and more than what x approaches. If the results are not very close, then the limit **does not exist**.

When you plug in what x approaches and get a whole number over 0, there is a **vertical asymptote**.

When you plug in what x approaches and get 0 over 0, there is a **removable discontinuity**.

Try to **factor and simplify** when there is a removable discontinuity.

A limit at a vertical asymptote is always **±∞**, or it does not exist.

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Congratulations

You have successfully completed
the core tutorial

Limits

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What's Next …

Step 1: Concepts – Core Tutorial (Just Completed)

→ **Step 2: Practice – Interactive Problem Drill**

Step 3: Recap – Super Review Cheat Sheet

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