16: Riemann Sums and the Definite Integral

Key Terms

Area Problem: Problem consisting in finding the area of a region of the xy-plane.
Partition of [*a*, *b*]: Finite and strictly increasing set of number such that the first point coincides with a and the last point coincides with b.
Uniform Partition: Partition in which consecutive points are equidistant from each other.
Sample Point: Point chosen between two consecutive points of a partition of [*a*, *b*].
Riemann Sum: Consider a function f continuous on [*a*,*b*]. Fill the region under the graph of f with rectangles. The corresponding Riemann sum is the some of the areas of the triangles above the *b*-axis and the negative of the areas of the rectangles below the *b*-axis.
Definite Integral of *f* from *a* to *b*: Limit of the Riemann sum of f as the number of rectangles approaches infinity.
Average of a Continuous Function f over [*a*,*b*]: Definite integral of f from a to b over b-a.

Key Formulas

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x, \text{ where } \Delta x = \frac{b-a}{n}.$$

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x, \text{ where } \Delta x = \frac{b-a}{n}.$$

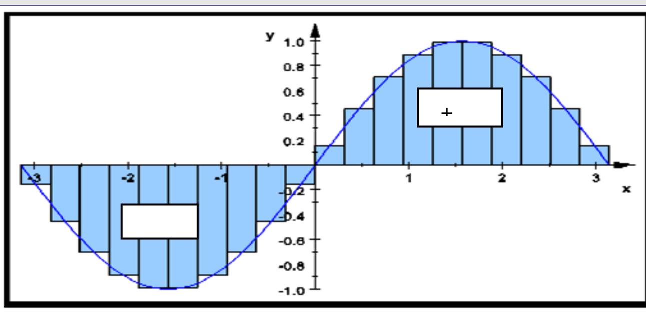
$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x, \text{ where } \Delta x = \frac{b-a}{n}.$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Variables Used

f = generic name given to a function continuous on $[a,b]$.
n = number rectangles, n+1=number of points of the subdivision
x_i = generic point of a partition of $[a,b]$.
x_i^* = generic name of a sample point.
S_n = Riemann sum; L_n = Left Riemann sum.
R_n = Right Riemann sum; M_n = Middle Riemann sum.

Riemann Sum



Properties of the Definite Integral

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b \big(f(x)-g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx, \text{ where } \alpha \text{ is a fixed number.}$$

$$\int_a^b \alpha\,dx = \alpha(b-a), \text{ where } \alpha \text{ is a constant.}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \text{ where } a \le c \le b.$$

If $g(x) \le f(x) \le h(x)$ for all x in $[a,b]$, then

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx.$$

Tips on Computing Riemann Sums and Manipulating Definite Integrals

- Calculate the points of the subdivision.
- Select the sample points.
- Apply the appropriate formula.
- Memorize the properties of definite integrals.

Typical Problems Riemann Sums and The Definite Integral

Example1: Find the left, right and middle Riemann sums of $f(x) = \sqrt{x} - 2, 1 \le x \le 6$ by using n=5 rectangles.
Answer: The uniform partition of [1, 6] is p={1,2,3,4,5,6}.

$$L_4 = -1 + (\sqrt{2}-2) + (\sqrt{3}-2) + (\sqrt{4}-2) + (\sqrt{5}-2) \approx -1.62$$

$$R_4 = (\sqrt{2}-2) + (\sqrt{3}-2) + (\sqrt{4}-2) + (\sqrt{5}-2) + (\sqrt{6}-2) \approx -0.17$$

$$M_4 = (\sqrt{1.5}-2) + (\sqrt{2.5}-2) + (\sqrt{3.5}-2) \ldots + (\sqrt{4.5}-2) + (\sqrt{5.5}-2) \approx -0.86$$

Example 2: If $\int_2^3 f(x)dx = -5$ and $\int_2^3 g(x)dx = 1$,

find $\int_2^3 [2f(x) + 3g(x)]dx$.

Answer2:

$$\int_2^3 [2f(x)+3g(x)]dx = \int_2^3 2f(x)dx + \int_2^3 3g(x)dx$$

$$= 2\int_2^3 f(x)dx + 3\int_2^3 g(x)dx$$

$$= 2(-5) + 3(1)$$

$$= -7$$

How to Use This Cheat Sheet: These are the keys related to this topic. Try to read through it carefully twice then recite it out on a blank sheet of paper. Review it again before the exams.

