

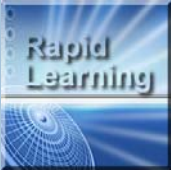
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 **Riemann Sums and
The Definite Integral**

Rapid Learning Tutorial Series

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Learning Objectives

By completing this tutorial, you will be able to ...



- Compute left, right and middle Riemann sums.
- Enunciate the definition of the definite integral.
- Provide the fundamental properties of the definite integral.
- Formulate the area problem in terms of the definite integral.

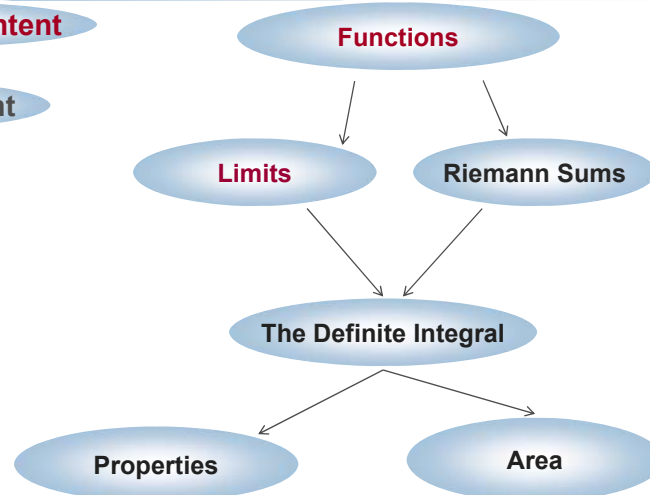
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Concepts Map

Previous content

New content

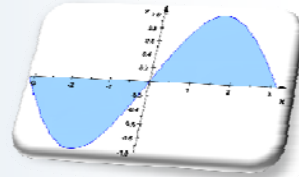


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The Area Problem



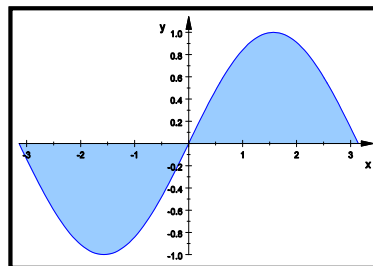
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Area of an Arbitrary Region of the Plane

The area problem consists of...

Finding the area of an arbitrarily-shaped region of the usual xy -plane.



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A Solution Strategy - 1

A possible approach is to **divide and conquer**.

We split the region into shapes for which the areas are known such as rectangles or triangles.



Then **add** the areas.

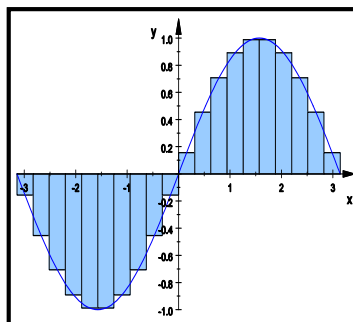
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A Solution Strategy - 2

If we have a perfect fit, then the problem is solved. Otherwise we obtain an **approximate solution**.

The divide-and-conquer approach fails to provide an **exact solution** when the boundary of the region is curved!



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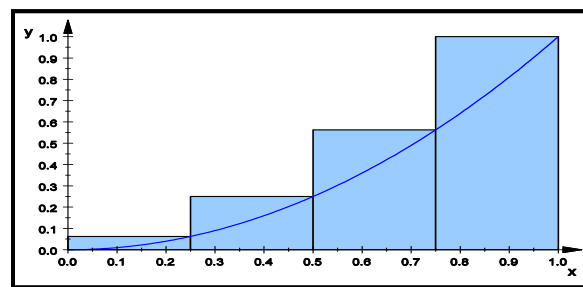


Example 1 (#1)

Approximate the area under the graph of $y = x^2$ from $x = 0$ to $x = 1$ by filling the region with four rectangles.

Splitting the interval $[0,1]$ into four parts gives endpoints 0.25, 0.5, 0.75, and 1 and each rectangle has width 0.25.

The **height** of each rectangle is given by the y values and are found to be $(0.25)^2$, $(0.5)^2$, $(0.75)^2$, and $(1)^2$.



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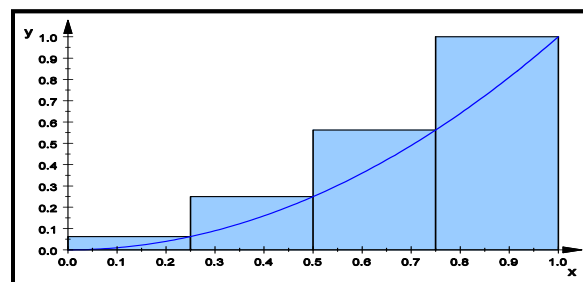


Example 1 (#2)

The total area of all of the rectangles is:

$$(0.25)(0.25)^2 + (0.25)(0.5)^2 + (0.25)(0.75)^2 + (0.25)(1)^2$$

$$\approx 0.4688$$



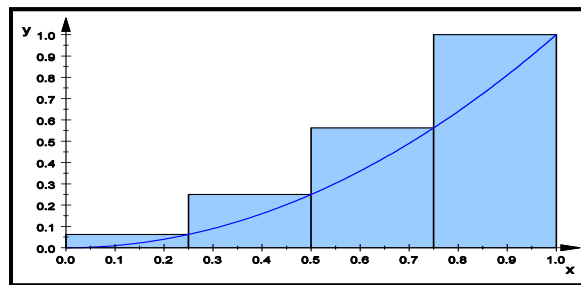
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Example 1 (#3)

Note that this is an **overestimate** of the area under the curve of $y = x^2$ from 0 to 1.



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Area Under the Graph



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Partition

To find the area under the graph, we first **partition** the interval into subintervals.

For example, splitting the interval $[0, 1]$ into 5 subintervals yields:

$$I_1 = [0, 0.2]$$

$$I_2 = [0.2, 0.4]$$

$$I_3 = [0.4, 0.6]$$

$$I_4 = [0.6, 0.8]$$

$$I_5 = [0.8, 1]$$



Note that 5 subintervals means that there are **6 points**.

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Area Under a Graph - 1

Next, we choose a sample point x_i^* within each subinterval.

We may choose the **left endpoint** of each subinterval



the **right endpoint**



or the **midpoint**



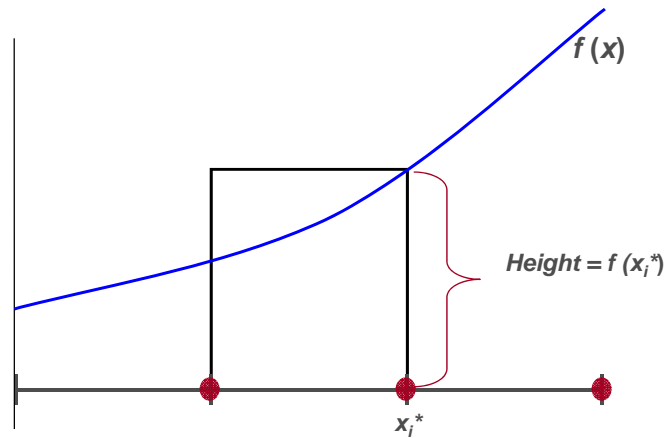
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Area Under a Graph - 2

The area of each rectangle is then the **width** of each subinterval multiplied by the **height** which is given by $f(x_i^*)$ for the i^{th} subinterval.



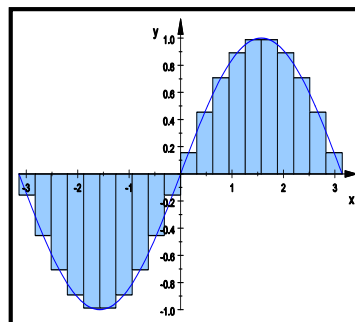
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Area Under a Graph - 3

Denoting the width of each subinterval by Δx , the sum of the area of all rectangles then becomes:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x$$



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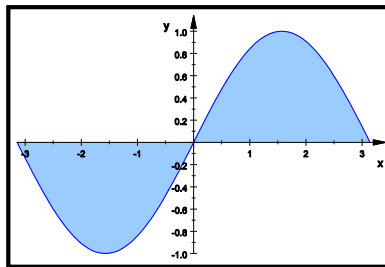




Area Under a Graph - 4

The more rectangles we have, the better the approximation. In fact, the **exact area** is given by letting n approach infinity. Therefore, the area under the curve is given by:

$$\lim_{n \rightarrow \infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x]$$



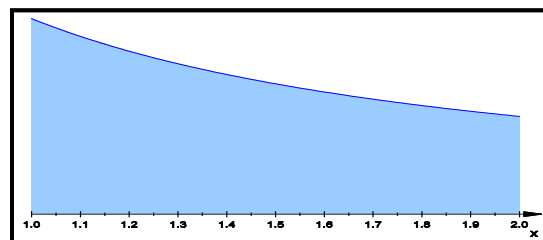
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Example – Problem Statement

Approximate the area under the graph of $y = 1/x$ from $x = 1$ to $x = 2$. Use a partition containing 5 points.

We work this problem using the **left** endpoints as the sample points, the **right** endpoints, and the **midpoints**.



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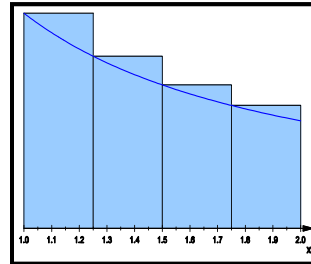


Example – Left Endpoints

The points of the partition are:

$$x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}, x_4 = 2.$$

Recall that since the function is $y = 1/x$, that the function value at each point is just the reciprocal.



Choosing the left endpoints, we obtain the following approximation:

$$L_4 = \left(\frac{1}{4}\right)(1) + \left(\frac{1}{4}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{319}{420} \approx 0.76$$

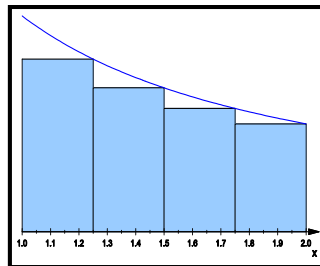
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Example – Right Endpoints

Choosing the right endpoints, we obtain the following approximation:

$$R_4 = \left(\frac{1}{4}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{4}{7}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{533}{840} \approx 0.63$$



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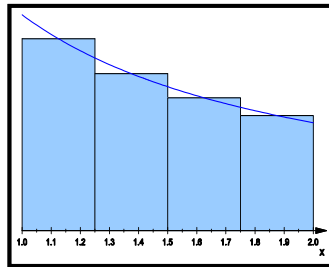




Example – Midpoints

Choosing the midpoints, we obtain the following approximation:

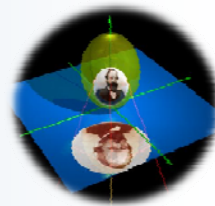
$$M_4 = \left(\frac{1}{4}\right)\left(\frac{8}{9}\right) + \left(\frac{1}{4}\right)\left(\frac{8}{11}\right) + \left(\frac{1}{4}\right)\left(\frac{8}{13}\right) + \left(\frac{1}{4}\right)\left(\frac{8}{15}\right) = \frac{4448}{6435} \approx 0.69$$



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Riemann Sums



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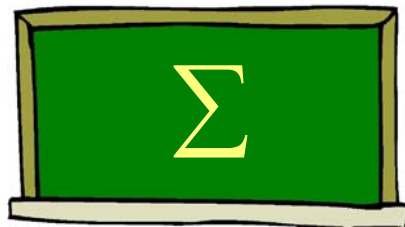




Preliminaries: Summation Notation - 1

This section shows how to:

Use sigma (or summation) notation to represent a sum.



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Preliminaries: Summation Notation - 2

Consider the numbers given by a_1, a_2, \dots, a_n

The i^{th} number is given by a_i

Summation notation provides a compact way to write sums such as:

$$a_1 + a_2 + \dots + a_n$$



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Preliminaries: Summation Notation - 3

The summation notation involves:

- 1 The summation sign
- 2 The variable which is being summed
- 3 The summation index
- 4 The lower limit of summation
- 5 The upper limit of summation



$$\sum_{i=1}^n a_i$$

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Preliminaries: Summation Notation - 4

The notation

$$\sum_{i=1}^n a_i$$

means to sum all of the terms beginning with a_1 and ending with a_n .

The variable i is called a **dummy variable** because the variable used is arbitrary.



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Properties of the Sigma Notation

Here are two properties of sigma notation.

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n ka_i = k \sum_{i=1}^n a_i$$



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Riemann Sums - 1

In this section, we use sigma notation to define **Riemann Sums**.



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Riemann Sums - 2

Suppose that we have a partition of the interval $[a, b]$ containing the points x_1, x_2, \dots, x_n .

Also, suppose that we have a sample point x_i^* within each subinterval $[x_{i-1}, x_i]$

The Riemann Sum is defined to be

$$S_n = \sum_{i=1}^n f(x_i^*) \cdot \Delta x$$

where $\Delta x = x_i - x_{i-1}$

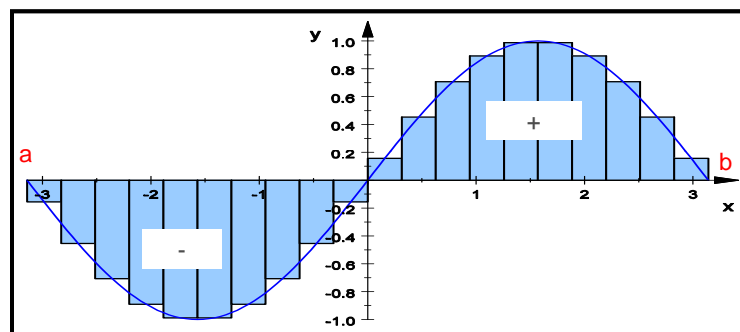


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Riemann Sums - 3

The Riemann Sum may be interpreted as the **areas of the rectangles** that lie above the x -axis and the negatives of the areas of the rectangles that are situated below the x -axis.



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The Left, Right, and Middle Riemann Sums

If the left endpoints of each subinterval are used as sample points, then we have a **Left Riemann Sum** denoted by L_n .

If the right endpoints of each subinterval are used as sample points, then we have a **Right Riemann Sum** denoted by R_n .

If the midpoints of each subinterval are used as sample points, then we have a **Middle Riemann Sum** denoted by M_n .



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Note – Riemann Sums

Riemann Sums
may be **Positive**,
Negative, or
Zero.

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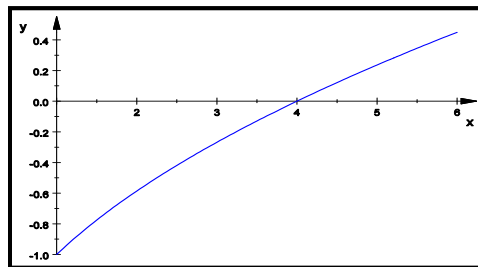




Example – Statement of the Problem

Find the left, right, and middle Riemann Sums using $n = 5$ where

$$f(x) = \sqrt{x} - 2, 1 \leq x \leq 6$$



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Example - Solution

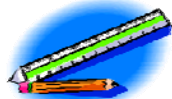
The partition of the interval $[1, 6]$, contains the values $\{1, 2, 3, 4, 5, 6\}$.

We find that:

$$L_4 = -1 + (\sqrt{2} - 2) + (\sqrt{3} - 2) + (\sqrt{4} - 2) + (\sqrt{5} - 2) \approx -1.62$$

$$R_4 = (\sqrt{2} - 2) + (\sqrt{3} - 2) + (\sqrt{4} - 2) + (\sqrt{5} - 2) + (\sqrt{6} - 2) \approx -0.17$$

$$M_4 = (\sqrt{1.5} - 2) + (\sqrt{2.5} - 2) + (\sqrt{3.5} - 2) + (\sqrt{4.5} - 2) + (\sqrt{5.5} - 2) \approx -0.86$$

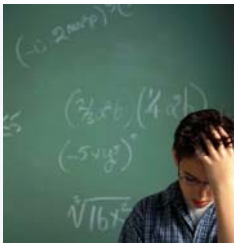


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The Definite Integral




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Definite Integral - 1

The **definite integral** of a function f from a to b is the limit of the Riemann Sum as the number of points in the partition approaches infinity.

The definite integral is denoted by:



$$\int_a^b f(x) dx$$

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Definite Integral - 2

In other words, if S_n is the Riemann Sum of f over $[a, b]$, then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} S_n$$

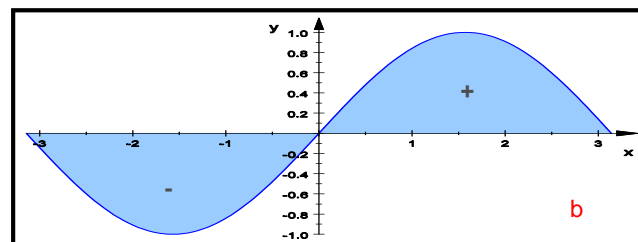


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Definite Integral - 3

The definite integral can be interpreted as the sum of the areas of the regions lying **above** the x axis and the negatives of the areas lying **below** the x axis.



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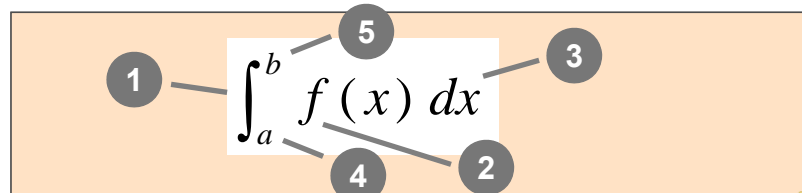




Definite Integral - 4

The definite integral involves

- 1 The integral sign
- 2 The integrand
- 3 The differential
- 4 The lower limit of integration
- 5 The upper limit of integration



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Properties of the Definite Integral - 1

Here are some **properties** of the definite integral (all c 's are assumed to be constants):

Linearity Property

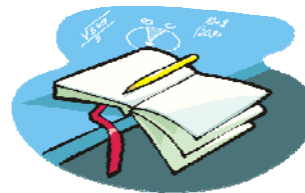
$$\int_a^b c_1 \cdot f(x) \pm c_2 \cdot g(x) dx = c_1 \int_a^b f(x) dx \pm c_2 \int_a^b g(x) dx$$

Constant Function Property

$$\int_a^b c dx = c(b - a)$$

Integral from a to a :

$$\int_a^a f(x) dx = 0$$



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Properties of the Definite Integral - 2

Here are some **properties** of the definite integral (all c 's are assumed to be constants):

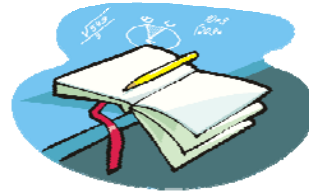
Splitting Property

If $a \leq b \leq c$ then

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

Reversing Limits Property:

$$\int_a^b f(x) dx = -\int_b^a f(x) dx$$



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Example – Definite Integral

Given that

$$\int_2^3 f(x) dx = -5 \text{ and } \int_2^3 g(x) dx = 1$$

Find

$$\int_2^3 [2f(x) + 3g(x)] dx.$$

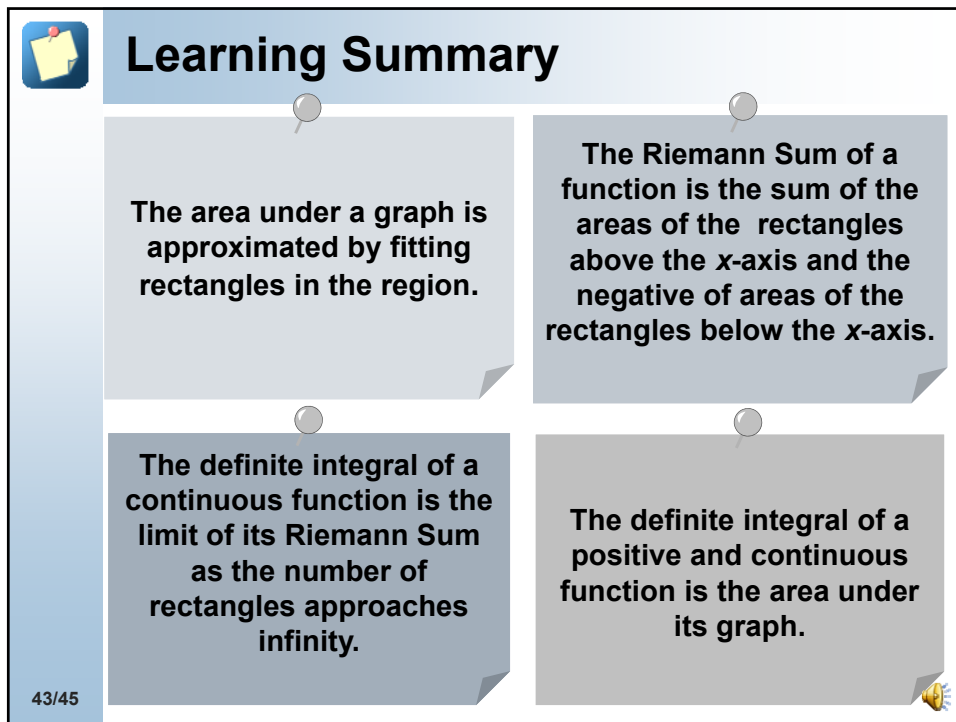
Solution:

$$\begin{aligned} \int_2^3 [2f(x) + 3g(x)] dx &= 2 \int_2^3 f(x) dx + 3 \int_2^3 g(x) dx \\ &= 2(-5) + 3(1) = -7 \end{aligned}$$



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Learning Summary

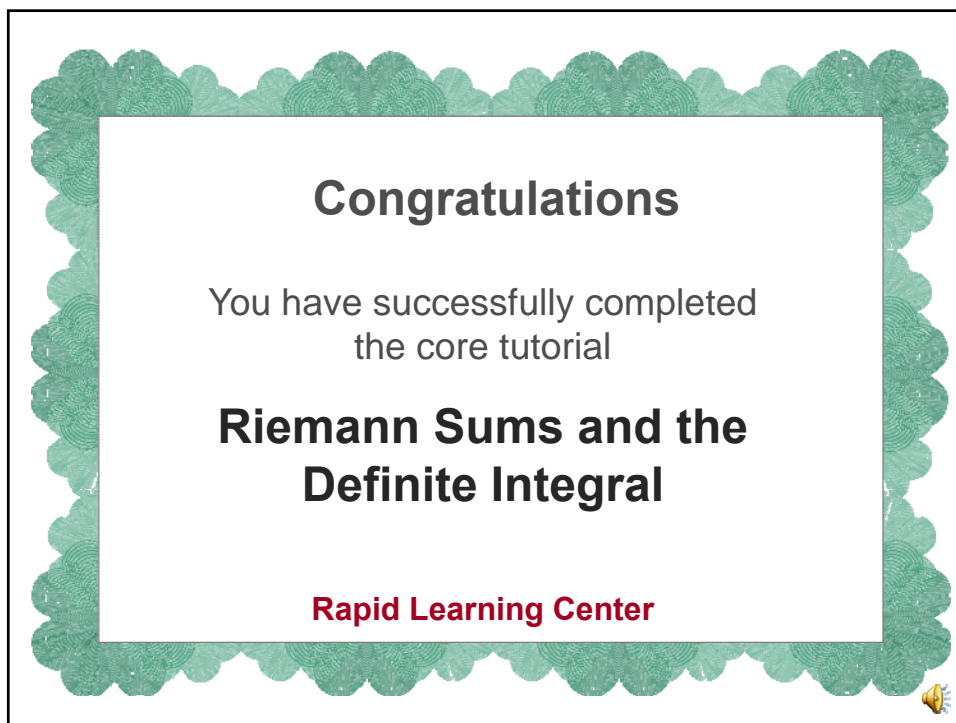
The area under a graph is approximated by fitting rectangles in the region.

The Riemann Sum of a function is the sum of the areas of the rectangles above the x -axis and the negative of areas of the rectangles below the x -axis.

The definite integral of a continuous function is the limit of its Riemann Sum as the number of rectangles approaches infinity.

The definite integral of a positive and continuous function is the area under its graph.

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Congratulations

You have successfully completed
the core tutorial

**Riemann Sums and the
Definite Integral**

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What's Next ...

Step 1: Concepts – Core Tutorial (Just Completed)

→ Step 2: Practice – Interactive Problem Drill

Step 3: Recap – Super Review Cheat Sheet

Go for it!



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