17: Iterated and Double Integrals

Integrating a Two Variable Function

Integrating a two variable function with respect to x yields a one variable function of y. Similarly, integrating a two variable function with respect to y will yield a one variable function of x.

Example: Evaluate $\int_1^4 2xy + 3y\,dx$

Solution: We have

$$\int_1^4 2xy + 3y\,dx = \left(x^2y + 3yx\right)\Big|_1^4$$
$$= 4^2y + 3y(4) - \left(1^2y + 3y(1)\right)$$
$$= 24y$$

Iterated Integrals

It is possible to integrate $f(x,y)$ with respect to one variable and then integrate again with respect to the other variable. This is called an **iterated integral**.

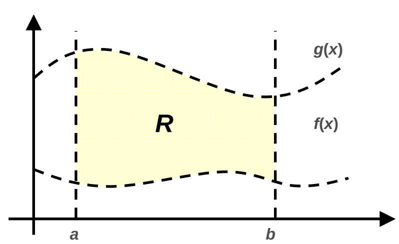
Example: Evaluate $\int_0^2\int_1^4 2xy + 3y\,dxdy$.

Solution: We have

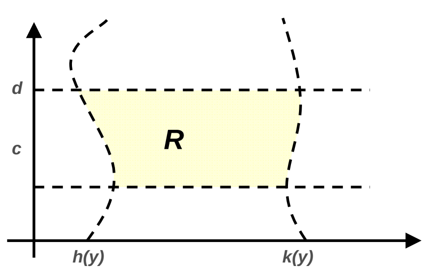
$$\int_0^2\int_1^4 2xy + 3y\,dxdy = \int_0^2 24y\,dy$$
$$= 12y^2\Big|_0^2$$
$$= 12(2)^2 - 12(0)^2$$
$$= 48$$

Area Between Two Curves

Given curves $f(x)$ and $g(x)$ over $[a,b]$, the area between the curves is given by $\int_a^b\int_{f(x)}^{g(x)} 1\,dydx$



Given curves $h(y)$ and $k(y)$ over $[c,d]$, the area between the curves is given by $\int_{h(y)}^{k(y)}\int_c^d 1\,dydx$



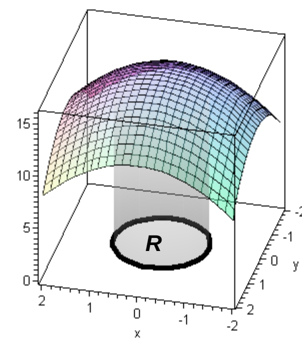
Order of Integration

The order in which the integration is performed may be interchanged, that is

$$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_c^d\int_a^b f(x,y)\,dy\,dx$$

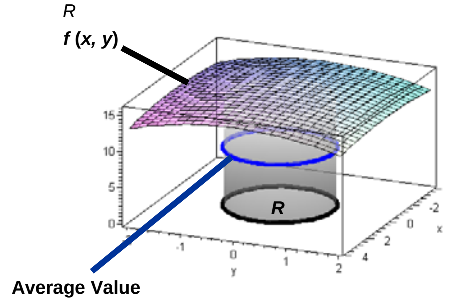
Volume

Given a function $f(x,y)$ and a region R, if $f(x,y) \geq 0$ over R then $\iint_R f(x,y)\,dA$ gives the volume under the surface.



Average Value

The average value of a function $f(x,y)$ over the region R is defined as $\frac{1}{A}\iint_R f(x,y)\,dA$ where A is the area of R.



Fubini's Theorem

The fact that the value of a double integral over the region R is the same no matter which order in which the integration is done is known as **Fubini's Theorem**.

Properties of Double Integrals

1. $\iint_R c \cdot f(x,y)dA = c\iint_R f(x,y)dA$
2. $\iint_R f(x,y) \pm g(x,y)\,dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$
3. $\iint_R f(x,y)dA \geq 0$ if $f(x,y) \geq 0$
4. $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$ if $f(x,y) \geq g(x,y)$

How to Use This Cheat Sheet: These are the keys related this topic. Try to read through it carefully twice then recite it out on a blank sheet of paper. Review it again before the exams.