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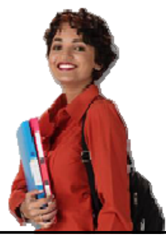

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Iterated and Double Integrals

Rapid Learning Tutorial Series

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Learning Objectives

By completing this tutorial, you will:

- Learn what iterated and double integrals are.
- Learn how to evaluate iterated and double integrals.
- Use double integrals to calculate area and volume.
- Apply double integrals to real-world phenomena.

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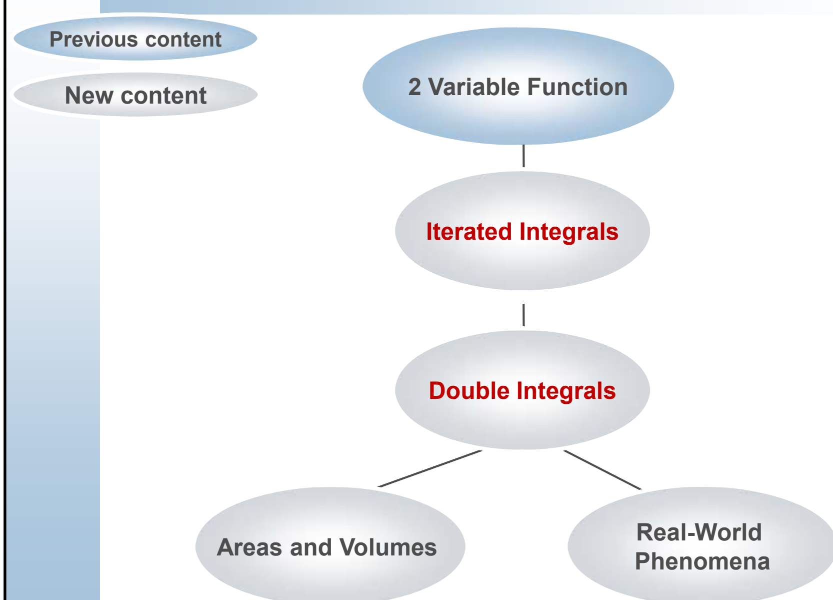
Concept Map



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Preliminaries

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Integrating a Two-Variable Function

Suppose that you are given a two variable function $f(x,y)$.

The process of integration is a one-variable process, that is, it can only be done with respect to one variable at a time.

Say we integrate with respect to x over a to b.

$$\int_a^b f(x,y)\,dx$$

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Integrating a Two-Variable Function (Cont. 1)

When we integrate with respect to *x*, we arrive at another function that involves both *x* and *y*.

However, by the Fundamental Theorem of Calculus, we then plug the values of *a* and *b* in place *x* in that function.

The result is a function that only involves *y*.

$$\int_a^b f(x,y)\,dx = F(y)$$

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Integrating a Two-Variable Function (Cont. 2)

Therefore, integrating a two variable function with respect to *x* yields a one variable function of *y*.

$$\int_a^b f(x,y)\,dx = F(y)$$

Similarly, integrating a two variable function with respect to *y* will yield a one variable function of *x*.

$$\int_a^b f(x,y)\,dy = F(x)$$

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Integrating a 2 Variable Function – Example

Example: Evaluate $\int_1^4 2xy + 3y\,dx$.

Solution:
Remember that, just like when differentiating with respect to one variable, the other variable(s) are treated like constants, the same is true with integrating with respect to one variable. We have:

$$\int_1^4 2xy + 3y\,dx = \left(x^2y + 3yx\right)\Big|_1^4$$

$$= 4^2y + 3y(4) - \left(1^2y + 3y(1)\right)$$

$$= 24y$$

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Last Example!

Note how, in the last example, when 2xy +3y is integrated with respect to *x*, we get a function only involving *y*.

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Iterated Integrals

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The Integral of an Integral

Previously, you saw an example of integrating *f*(*x*,*y*) with respect to one variable.

It is possible to integrate *f*(*x*,*y*) with respect to one variable and then integrate again with respect to the other variable.

This is called an iterated integral.

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

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Notation

The standard notation omits the parentheses when writing **iterated integrals**.

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Iterated Integral – Example

Example: Evaluate $\int_0^2 \int_1^4 2xy + 3y \, dxdy$.

Solution:
Recall from the previous example that

$$\int_1^4 2xy + 3y \, dx = 24y$$

therefore,

$$\int_0^2 \int_1^4 2xy + 3y \, dxdy = \int_0^2 24y \, dy$$

$$= 12y^2 \Big|_0^2$$

$$= 12(2)^2 - 12(0)^2$$

$$= 48$$

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Area

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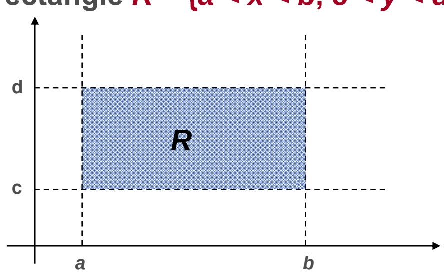
Area

Consider a rectangle $R = \{a < x < b,\ c < y < d\}$.



The area of the rectangle is easily found to be $(d - c)(b - a)$.

Now, note that

$$\int_a^b \int_c^d 1\, dydx = \int_a^b y\Big|_c^d dx = \int_a^b (d - c)\, dx$$
$$= (d - c)x\Big|_a^b = (d - c)(b - a)$$

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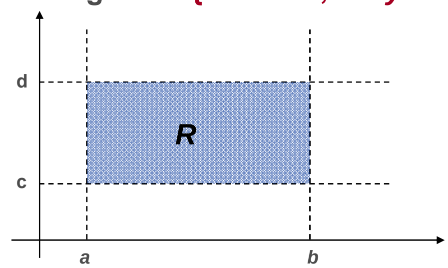
Area (Cont. 1)

Consider a rectangle $R = \{a < x < b, c < y < d\}$.



That is, an iterated integral where $f(x,y) = 1$, yields the **area** of the region R.

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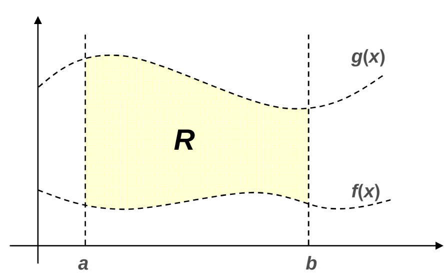
Area (Cont. 2)

The previous result can be generalized to find the area between any **two curves**.

Given curves $f(x)$ and $g(x)$ over $[a,b]$, the area between the curves is given by

$$\int_a^b \int_{f(x)}^{g(x)} 1 \, dydx$$



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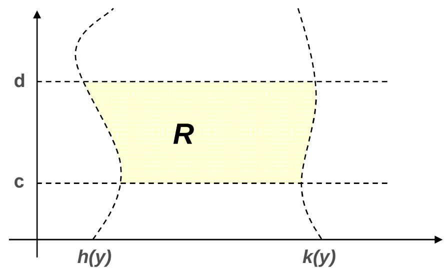
Area (Cont. 3)

Similarly, the area between two functions of *y* can be found:

Given curves *h*(*y*) and *k*(*y*) over [*c*,*d*], the area between the curves is given by

$$\int_{h(y)}^{k(y)} \int_{c}^{d} 1\, dydx$$



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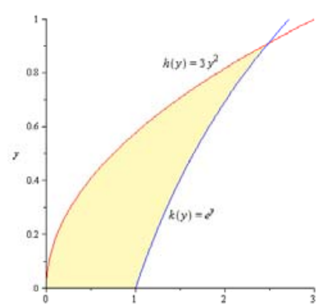
Area – Example 1

Example:
Use an iterated integral to find the area of the region bounded by $h(y) = 3y^2$ and $k(y) = e^y$ over [0,1].



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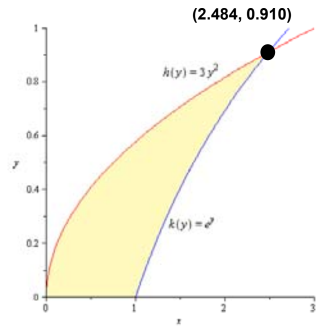
Area – Example 1 (Cont.)

Solution:

First, we find the intersection point of the two graphs:

$$3y^2 = e^y$$

Using a calculator, we find the point of intersection to be **(2.484, 0.910)**.



Now, we set up and evaluate the integrals:

$$\int_0^{0.910}\int_{3y^2}^{e^y} 1\,dx\,dy = \int_0^{0.910} x\Big|_{3y^2}^{e^y} dy = \int_0^{0.910} e^y - 3y^2\,dy$$

$$= \left(e^y - y^3\right)\Big|_0^{0.910} = e^{0.910} - (0.910)^3 - \left(e^0 - 0^3\right) \approx 1.731 - 1 = 0.731$$

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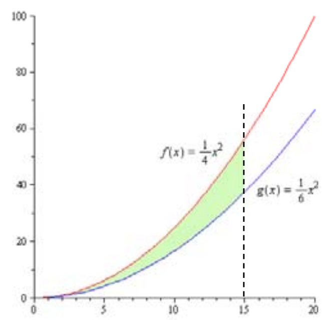
Area – Example 2

Example:

Suppose that Car A is traveling at $f(x) = 1/4x^2$ ft/s and that Car B is traveling along the same road at $g(x) = 1/6x^2$ ft/s.

The **area** between $f(x)$ and $g(x)$ represents the **distance** between the faster car (Car A) and the slower car (Car B).

Find the distance between the cars after 15 seconds.



Solution:

We set up the integrals:

$$\int_0^{15}\int_{1/6x^2}^{1/4x^2} 1dy\,dx$$

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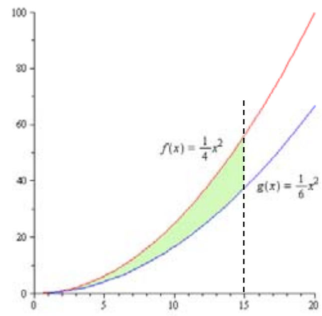
Area – Example 2 (Cont.)

We have:

$$\int_0^{15}\int_{1/6x^2}^{1/4x^2} 1\,dy\,dx = \int_0^{15} y\Big|_{1/6x^2}^{1/4x^2} dx = \int_0^{15}\left(\frac{1}{4}x^2 - \frac{1}{6}x^2\right)dx$$

$$= \int_0^{15}\frac{1}{12}x^2 dx = \frac{1}{36}x^3\Big|_0^{15} = \frac{1}{36}(15)^3 - \frac{1}{36}(0)^3 = \frac{375}{4} = 93.75$$



After 15 seconds, the cars are 93.75 feet apart.

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Order of Integration

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Different Orders of Integration

The order in which the integration is performed may be **interchanged**.

Changing the order of integration may make the process harder or easier.

The value of the integral is not affected.

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$$

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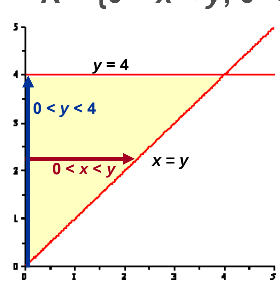
Order of Integration – Example

Example: Interchange the order of integration and find the value of the iterated integral.

$$\int_0^4 \int_0^y f(x,y)\,dx\,dy$$

Solution: First note that the region over which we are integrating is:

$R = \{0 < x < y,\ 0 < y < 4\}$



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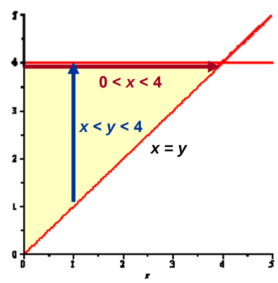
Order of Integration – Example (Cont.)

Solution: Note that the shaded region may also be written as:

$$R = \{ 0 < x < 4,\ x < y < 4 \}$$

Therefore, we may write

$$\int_0^4 \int_0^y f(x,y)\,dx\,dy = \int_0^4 \int_x^4 f(x,y)\,dy\,dx$$



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Volume

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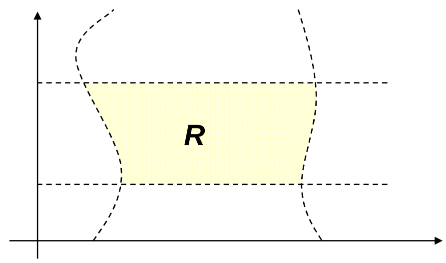
From Area to Volume

When evaluating an integral over an **interval**, the result is the **area** of a region.

When evaluating an iterated integral over a **region**, the result is the **volume** under a surface.



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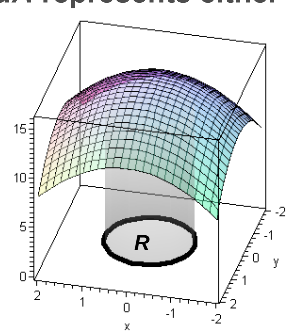
Formula for Volume

Given a function $f(x,y)$ and a region R, if $f(x,y) \geq 0$ over R then

$$\iint_R f(x,y)\,dA$$

gives the **volume** under the surface.

The differential dA represents either $dxdy$ or $dydx$.



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Double Integrals

Integrating over a general region requires two integrals and are thus called **double integrals**.

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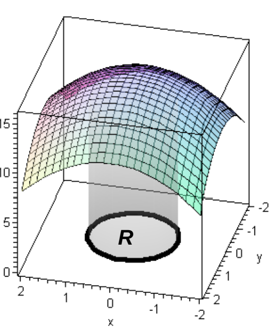
Volume - Example

Example: Find the volume of the solid under the surface of $f(x,y) = 16 - x^2 - y^2$ over the unit circle in the xy plane.



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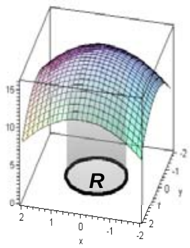
Volume – Example (Cont. 1)

Solution: First note that the unit circle in the *xy* plane has equation $x^2 + y^2 = 1$.



The bounds on x are:

$$x = -\sqrt{1-y^2}$$

and

$$x = \sqrt{1-y^2}$$

The bounds on y are –1 and 1.

That is,

$$R = \{-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, -1 \le y \le 1\}$$

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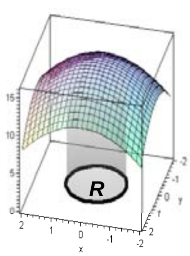
Volume – Example (Cont. 2)

Solution: Now we set up and solve the double integral:

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(16 - x^2 - y^2\right)dx\,dy$$



$$= \int_{-1}^{1}\left(16x - \frac{1}{3}x^3 - y^2x\right)\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1}\left(\frac{94}{3}\sqrt{1-y^2} - \frac{4}{3}y^2\sqrt{1-y^2}\right)dy$$

This integral can be evaluated analytically, but is somewhat complicated.

It is generally easier to evaluate the integral numerically.

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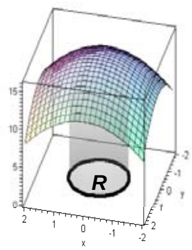
Volume – Example (Cont. 3)

Solution: Continuing analytically, we find:



$$\int_{-1}^{1}\left(\frac{94}{3}\sqrt{1-y^2}-\frac{4}{3}y^2\sqrt{1-y^2}\right)dy$$

$$=\frac{31}{2}y\sqrt{1-y^2}+\frac{31}{2}\sin^{-1}(y)+\frac{1}{3}y(1-y^2)^{3/2}\Big|_{-1}^{1}$$

$$=\frac{31\pi}{2}\approx 48.6947$$

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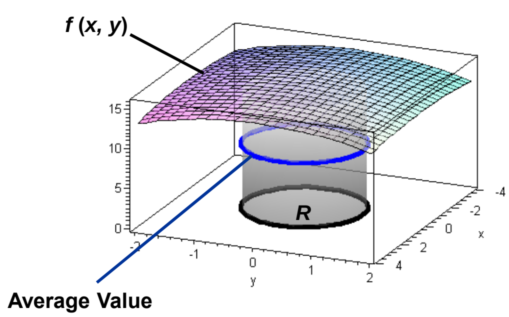
Average Value

The **average value** of a function $f(x,y)$ over the region R is defined as

$$\frac{1}{A}\iint_R f(x,y)\,dA$$

where A is the area of R.



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Average Value – Example

Example:

The production level of a company is given by $f(x,y) = 200x^{0.7}y^{0.3}$ where x is the number of units of labor and y is the number of units of capital.

Find the average production level if the number of units of labor varies between 100 and 150 and the number of units of capital varies between 325 and 400.

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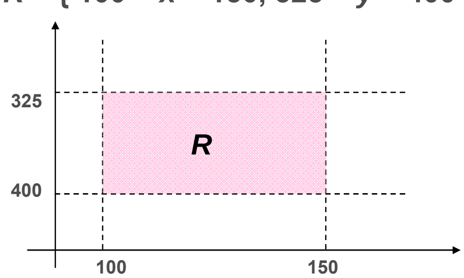
Average Value – Example (Cont. 1)

Solution:

The region in the *xy* plane is defined by

$R = \{ 100 < x < 150,\ 325 < y < 400 \}$.



The area of the region R is easily found to be $A = 1250$.

Therefore, the average value is given by

$$\frac{1}{A}\iint_R f(x,y)\,dA = \frac{1}{1250}\int_{100}^{150}\int_{325}^{400}\left(200x^{0.7}y^{0.3}\right)dy\,dx$$

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Average Value – Example (Cont. 2)

Solution (Continued):

We have:

$$\frac{1}{1250}\int_{100}^{150}\int_{325}^{400}\left(200x^{0.7}y^{0.3}\right)dy\,dx$$

$$=\frac{1}{1250}\int_{100}^{150}\left(153.8462x^{0.7}y^{1.3}\right)\Big|_{325}^{400}dx$$

$$=\frac{1}{1250}\int_{100}^{150}87845.7113x^{0.7}dx$$

$$=\frac{1}{1250}\left(51673.9478x^{1.7}\right)\Big|_{100}^{150}$$

$$=\frac{1}{1250}\left(51673.9478(150)^{1.7}-51673.9478(100)^{1.7}\right)$$

$$\approx 103{,}040$$

The **average production level** is 103,040 units

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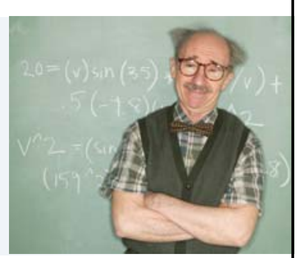
Additional Remarks



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Fubini's Theorem

The fact that the value of a double integral over the region *R* is the same no matter which order in which the integration is done is known as **Fubini's Theorem**.

It is named after the the Italian mathematician Guido Fubini.

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Properties of Double Integrals

Many of the **properties** of double integrals mirrors those of single integrals. Specifically,

1. $\iint_R c \cdot f(x,y)dA = c\iint_R f(x,y)dA$

2. $\iint_R f(x,y) \pm g(x,y)dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$

3. $\iint_R f(x,y)dA \geq 0 \text{ if } f(x,y) \geq 0$

4. $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA \quad \text{if } f(x,y) \geq g(x,y)$

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Learning Summary

Volume and Average Value are **applications** of double integrals.

An **Iterated Integral** is an integral within an integral.

Iterated integrals can be used to find **Area**.

The order of integration can be **interchanged** but one order may be easier than the other.

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Congratulations

You have successfully completed the core tutorial

Iterated and Double Integrals

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→ Step 2: Practice – Interactive Problem Drill
Step 3: Recap – Super Review Cheat Sheet
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