## 16: Riemann Sums and the Definite Integral

### Key Terms

**Area Problem:** Problem consisting in finding the area of a region of the xy-plane.

**Partition of** \([a, b]\): Finite and strictly increasing set of number such that the first point coincides with \(a\) and the last point coincides with \(b\).

**Uniform Partition:** Partition in which consecutive points are equidistant from each other.

**Sample Point:** Point chosen between two consecutive points of a partition of \([a, b]\).

**Riemann Sum:** Consider a function \(f\) continuous on \([a, b]\).

- Fill the region under the graph of \(f\) with rectangles. The corresponding Riemann sum is the sum of the areas of the triangles above the \(b\)-axis and the negative of the areas of the rectangles below the \(b\)-axis.

**Definite Integral of** \(f\) **from** \(a\) **to** \(b\): Limit of the Riemann sum of \(f\) as the number of rectangles approaches infinity.

**Average of a Continuous Function** \(f\) **over** \([a, b]\): Definite integral of \(f\) from \(a\) to \(b\) over \(b-a\).

### Key Formulas

- \(S_n = \sum_{i=1}^{n} f(x_i^*) \Delta x\), where \(\Delta x = \frac{b-a}{n}\).
- \(L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x\), where \(\Delta x = \frac{b-a}{n}\).
- \(M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x\), where \(\Delta x = \frac{b-a}{n}\).
- \(\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x\).

### Properties of the Definite Integral

- \(\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx\)
- \(\int (f(x) - g(x)) \, dx = \int f(x) \, dx - \int g(x) \, dx\)
- \(\int a f(x) \, dx = a \int f(x) \, dx\), where \(a\) is a fixed number.
- \(\int a f(x) \, dx = a(b - a)\), where \(a\) is a constant.
- \(\int f(x) \, dx = c\int f(x) \, dx + c\int f(x) \, dx\), where \(a \leq c \leq b\).
- If \(g(x) \leq f(x) \leq h(x)\) for all \(x\) in \([a, b]\), then
  \(\int g(x) \, dx \leq \int f(x) \, dx \leq \int h(x) \, dx\).

### Tips on Computing Riemann Sums and Manipulating Definite Integrals

- Calculate the points of the subdivision.
- Select the sample points.
- Apply the appropriate formula.
- Memorize the properties of definite integrals.

### Typical Problems Riemann Sums and The Definite Integral

**Example 1:** Find the left, right and middle Riemann sums of \(f(x) = \sqrt{x} - 2, 1 \leq x \leq 6\) by using \(n=5\) rectangles.

**Answer:** The uniform partition of \([1, 6]\) is \(p = \{1, 2, 3, 4, 5, 6\}\).

- \(L_5 = -1 + (\sqrt{2} - 2) + (\sqrt{3} - 2) + (\sqrt{4} - 2) + (\sqrt{5} - 2) \approx -1.62\)
- \(R_5 = (\sqrt{2} - 2) + (\sqrt{3} - 2) + (\sqrt{4} - 2) + (\sqrt{5} - 2) + (\sqrt{6} - 2) \approx -0.17\)
- \(M_5 = (\sqrt{1.5} - 2) + (\sqrt{2.5} - 2) + (\sqrt{3.5} - 2) + \ldots + (\sqrt{5.5} - 2) \approx -0.86\)

**Example 2:** If \(\int_{2}^{3} f(x) \, dx = -5\) and \(\int_{2}^{3} g(x) \, dx = 1\), find \(\int_{2}^{3} [2f(x) + 3g(x)] \, dx\).

**Answer:**

\[
\int_{2}^{3} [2f(x) + 3g(x)] \, dx = \int_{2}^{3} 2f(x) \, dx + \int_{2}^{3} 3g(x) \, dx
\]

\[
= 2 \int_{2}^{3} f(x) \, dx + 3 \int_{2}^{3} g(x) \, dx
\]

\[
= 2(-5) + 3(1)
\]

\[
= -7
\]

### How to Use This Cheat Sheet

These are the keys related to this topic. Try to read through it carefully twice then recite it out on a blank sheet of paper. Review it again before the exams.